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Localizing Volatilities*

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Abstract

We propose two main applications of Gyöngy (1986)'s construction of inhomogeneous Markovian stochastic differential equations that mimick the one-dimensional marginals of continuous Itô processes. Firstly, we prove Dupire (1994) and Derman and Kani (1994)'s result. We then present Bessel-based stochastic volatility models in which this relation is used to compute analytical formulas for the local volatility. Secondly, we use these mimicking techniques to extend the well-known local volatility results to a stochastic interest rates framework.

1 Introduction

It has been widely accepted for at least a decade that the option pricing theory of Black and Scholes (1973) and Merton (1973) has been inconsistent with option prices. Actually, the model implies that the informational content of the option surface is one dimensional which means that one could construct the prices of options at all strikes and maturities from the price of any single option. It has also been shown that unconditional returns show excess kurtosis and skewness which are inconsistent with normality. Special attention was given to implied volatility smile or skew, but research has concentrated on implied Black and Scholes volatility since it has become the unique way to price vanilla options. Accordingly, option prices are often quoted by their implied volatility. Nevertheless, this method is unsuitable for more complicated exotic options and options with early exercise features. To explain in a self-consistent way why options with different strikes and maturities have different implied volatilities or what one calls the volatility smile, one could use stochastic volatility models (eg. Heston (1993) or Hull and White (1987))

Given the computational complexity of stochastic volatility models and the difficulty of fitting their parameters to the market prices of vanilla options,

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practitioners found a simpler way to price exotic options consistently with the volatility smile by using local volatility models as introduced by Dupire (1994) and Derman and Kani (1994). Local volatility models have the advantage to fit the implied volatility surface; hence, when pricing an exotic option, one feels comfortable hedging through the stock and vanilla options markets.

In the last twenty years, academics and practitioners have been primarily interested in building models that describe well the behavior of an asset whether it is equity, FX, Credit, Fixed-Income or Commodities and very rarely models that specify any cross-asset dependency. For all cross-asset derivative products, this dependency modifies the model one should use or at least the calibration procedure. Certainly, models that incorporate a dependency on other asset classes than a specific underlying need to be recalibrated as soon as the other asset classes become random, in particular in the fast growing hybrid industry where it is necessary to model several assets.

The remainder of the paper is organized as follows. Section 2 recalls preliminary results on Bessel processes and states mimicking properties of continuous Itô processes exhibited by Gyöngy (1986) and Krylov (1985). Section 3 recalls well-known results of Dupire (1994) and Derman and Kani (1994) on local volatility, gives a proof of the existence of a local volatility model that mimicks a stochastic volatility one based on Gyöngy (1986) theorem. Section 4 gives examples of stochastic volatility models where a local volatility can be computed. Those examples are based on remarkable properties of Bessel processes such as scaling properties. In order to extend the class of volatility models (where closed-form formulas can be obtained), we propose a general framework in which the volatility diffusion is a general deterministic time and space transformation of Bessel processes. Analytical computations are proposed in cases where the volatility diffusion is independent from the stock price diffusion as well as in cases where they are correlated. Section 5 applies the results of Section 3 to the case of stochastic interest rates and more generally shows how Gyöngy (1986) theorem can be applied to construct a local volatility model in a deterministic interest rate framework, starting from a stochastic volatility model with stochastic rates. Finally, Section 6 concludes our work and presents an important open question on mimicking the laws of Itô processes.

2 Preliminary Mathematical Results

2.1 Bessel and CIR Processes

Let $(R_t, t \geq 0)$ denote a Bessel process with dimension δ, starting from 0 and $(\beta_t, t \geq 0)$ an independent brownian motion from $(B_t, t \geq 0)$ the driving brownian motion. Let us recall that R_t^2 solves the following SDE:

$$dR_t^2 = 2R_t dB_t + \delta dt$$

and let us now define :

$$I_t = \int_0^t R_s d\beta_s \quad and \quad A_t = \int_0^t R_s^2 ds$$

Then, the one-dimensional marginals of (A_t, I_t) are at least in theory well-identified, via Fourier-Laplace expressions, and are closely related with the so-called Lévy area formula (see Lévy (1950), Williams (1976), Gaveau (1977), Yor (1980), Chapter 2 of Yor (1992) and many other references). Here we simply recall, for our purposes the formulae:
$\forall(\alpha, \beta) \in \mathbb{R}^2$

$$\mathbb{E}\Big[\exp\big(i\alpha I_t - \frac{\beta^2}{2} A_t\big)\Big] = \big(\cosh(t\sqrt{\alpha^2+\beta^2})\big)^{-\frac{\delta}{2}} \tag{1}$$

as well as:
$\forall(a, b) \in \mathbb{R}_+ \times \mathbb{R}$

$$\mathbb{E}\Big[\exp\big(-aR_t^2 - \frac{b^2}{2} A_t\big)\Big] = \big(\cosh(bt) + \frac{2a}{b}\sinh(bt)\big)^{-\frac{\delta}{2}} \tag{2}$$

a formula that we shall use later. Some developments for the law of A_t are given, e.g. in Pitman and Yor (2003).
For a Bessel process of dimension δ starting at x, one gets the following formula:
$\forall(a, b) \in \mathbb{R}_+ \times \mathbb{R}$

$$\mathbb{E}_x\Big[\exp\big(-aR_t^2 - \frac{b^2}{2} A_t\big)\Big] = \big(\cosh(bt) + \frac{2a}{b}\sinh(bt)\big)^{-\frac{\delta}{2}} \times \exp\Bigg(-\frac{x^2 b}{2}\frac{\sinh(bt) + \frac{2a}{b}\cosh(bt)}{\cosh(bt) + \frac{2a}{b}\sinh(bt)}\Bigg)$$

Let us now present a scaling property of the Bessel process with respect to conditioning, which is important in the sequel.

Proposition 2.1 *For any Bessel process R_t with dimension δ, we have:*

$$\mathbb{E}\Big[R_t^2 \big| \int_0^t R_s^2 ds\Big] = \frac{2}{t}\int_0^t R_s^2 ds \tag{3}$$

Remark 2.2 *This result is in fact a very particular case of a more general result involving only the scaling property of the process $(R_t^2, t \geq 0)$, see, e.g, Pitman and Yor (2003). But, for the sake of completeness, we shall give a direct proof of (3) below:*

Proof : From the scaling property of $(R_t^2, t \geq 0)$, we deduce that for every $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}_+)$, with bounded derivative, we have:

$$\mathbb{E}\Big[f\Big(\int_0^t R_s^2 ds\Big)\Big] = \mathbb{E}\Big[f\Big(t^2 \int_0^1 R_s^2 ds\Big)\Big]$$

We then differentiate both sides with respect to t, to obtain:

$$\begin{aligned}
\mathbb{E}\Big[f'\Big(\int_0^t R_s^2 ds\Big)R_t^2\Big] &= \mathbb{E}\Big[f'\Big(t^2\int_0^1 R_s^2 ds\Big)(2t)\int_0^1 R_s^2 ds\Big] \\
&= \mathbb{E}\Big[f'\Big(\int_0^t R_s^2 ds\Big)\frac{2}{t}\int_0^t R_s^2 ds\Big]
\end{aligned}$$

Since this identity is true for every bounded Borel function f', the identity (3) follows. ■

Remark 2.3 *We now check that formula (3) can be obtained directly as a consequence of formula (2): differentiating (2) both sides with respect to a and taking $a = 0$, we obtain:*

$$\mathbb{E}\Big[R_t^2 \exp\big(-\frac{b^2}{2}A_t\big)\Big] = \frac{\delta}{(\cosh(bt))^{\frac{\delta}{2}+1}}\big(\frac{1}{b}\sinh(bt)\big)$$

while, taking $a = 0$ in (2), and differentiating both sides with respect to b, we get:

$$b\mathbb{E}\Big[A_t \exp\big(-\frac{b^2}{2}A_t\big)\Big] = \frac{\delta t}{2(\cosh(bt))^{\frac{\delta}{2}+1}}\sinh(bt)$$

and the identity (2) follows from the comparison of these last two equations.

A reason why squared Bessel processes play an important role in financial mathematics is that they are connected to models used in finance. One of these models is the Cox, Ingersoll and Ross (1985) CIR family of diffusions which are solutions of the following kind of SDEs:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{|X_t|}dW_t \tag{4}$$

with $X_0 = x_0 > 0$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma > 0$ and W_t a standard brownian motion. This equation admits a unique strong (that is to say adapted to the natural filtration of W_t) solution that takes values in $\mathbb{R}_+$.

One is now interested in the representation of a CIR process in terms of a time-space transformation of a Bessel process:

Lemma 2.4 *A CIR Process X_t solution of equation (4) can be represented in the following form:*

$$X_t = e^{-bt}R^2_{\frac{\sigma^2}{4b}(e^{bt}-1)} \tag{5}$$

where R denotes a Bessel process starting from x_0 at time $t = 0$ of dimension $\delta = \frac{4a}{\sigma^2}$

Proof : This lemma results from simple properties of squared Bessel processes that can be found in Revuz and Yor (2001), Pitman and Yor (1980, 1982). ■

This relation has been widely used in finance, for instance in Geman and Yor (1993) or Delbaen and Shirakawa (1996).

2.2 Mimicking Theorems

A common topic of interest of Krylov and Gyöngy respectively in Krylov (1985) and Gyöngy (1986) is the construction of stochastic differential equations whose solutions mimick certain features of the solutions of Itô processes. The construction of Markov martingales that have specified marginals was studied by Madan and Yor (2002). Bibby, Skovgaard and Sørensen (2005) as well as Bibby and Sørensen (1995) proposed construction of diffusion-type models with given marginals.

Let us now consider an Itô differential equation of the form:

$$\xi_t = \int_0^t \delta_s dW_s + \int_0^t \beta_s ds \tag{6}$$

where W_t is a $\mathcal{F}_t$-Brownian motion of dimension k, $(\delta_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$ are bounded $\mathcal{F}_t$-adapted processes that belong respectively to $\mathbf{M}_{n,k}(\mathbb{R})$, the space of $n \times k$ real matrices and to $\mathbb{R}^n$.

Definition 2.5 (Green Measure) *Considering two stochastic processes X_t, valued in $\mathbb{R}^n$ and γ_t, with $\gamma_t > 0$, one defines the Green measure $\mu_{X,\gamma}$ by:*

$$\mu_{X,\gamma}(\Gamma) = \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(X_t)\exp\big(-\int_0^t \gamma_s ds\big)dt\Big] \tag{7}$$

where Γ is any borel set of $\mathbb{R}^n$

Remark 2.6 *The stochastic process γ_t is called the killing rate*

Theorem 2.7 (Krylov) *If ξ_t is an Itô process defined as previously and satisfying the uniform ellipticity condition: $\exists\lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda I_n$*
as well as the lower boundedness condition: $\exists \alpha \in \mathbb{R}_+$ such as $\gamma > \alpha$,
then there exist deterministic functions $\sigma : \mathbb{R}^n \to \mathbf{M}_n(\mathbb{R})$, $b : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}_+$ such that the following SDE:

$$\begin{aligned} dx_t &= \sigma(x_t)dW_t + b(x_t)dt \\ x_0 &= 0 \end{aligned}$$

has a weak solution satisfying:

$$\begin{aligned} &\forall\Gamma \in \mathcal{B}(\mathbb{R}^n) \\ \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(\xi_t)\exp\big(-\int_0^t \gamma_s ds\big)dt\Big] &= \mathbb{E}\Big[\int_0^\infty \mathbf{1}_\Gamma(x_t)\exp\big(-\int_0^t c(x_s)ds\big)dt\Big] \\ ie : \mu_{\xi,\gamma}(\Gamma) &= \mu_{X,c}(\Gamma) \end{aligned}$$

Proof : See Krylov (1985) ∎

Definition 2.8 (Weak Solution) *The stochastic differential equation*

$$dX_t = f(t, X_t)dW_t + g(t, X_t)dt \tag{8}$$

$$X_0 = 0 \tag{9}$$

is said to have a weak solution if there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathcal{F}_t$-Brownian motion with respect to which there exists a $\mathcal{F}_t$-adapted stochastic process $\overline{X}_t$ that satisfies (8) and (9).

A natural question asked and answered by Gyöngy is whether it is possible to find the solution of an SDE with the same one-dimensional marginal distributions as an Itô process. The answer is stated below:

Theorem 2.9 (Gyöngy) *If ξ_t is an Itô process satisfying the uniform ellipticity condition: $\exists \lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda I_n$ then there exist bounded measurable functions $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbf{M}_{n,n}(\mathbb{R})$ and $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ defined by:*

$$
\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n
$$
$$
\begin{aligned}
\sigma(t,x) &= \Big(\mathbb{E}\big[\delta_t\delta_t^* | \xi_t = x\big]\Big)^{\frac{1}{2}} \\
b(t,x) &= \mathbb{E}\big[\beta_t | \xi_t = x\big]
\end{aligned}
$$

such that the following SDE:

$$
\begin{aligned}
dx_t &= \sigma(t,x_t)dW_t + b(t,x_t)dt \\
x_0 &= 0
\end{aligned}
$$

has a weak solution with the same one-dimensional marginals as ξ.

Proof : See Gyöngy (1986) ■

Remark 2.10 *Two kinds of mimicking features of a general Itô process were illustrated in this section . With Krylov, we were able to construct a Markov homogeneous process solution of an SDE, that has the same Green measure than the Itô process. Using Gyöngy's results, we were able to build a time-inhomogeneous Markov process solution of an SDE that has the same one-dimensional marginals as the Itô process.*

A possible extension to the above mimicking property is to consider a real Itô process ξ driven by a multidimensional Brownian motion and obtain a new mimicking result useful for the remainder of the paper; the proof is straightforward from Gyöngy (1986) proof. Let ξ be as follows :

$$
\xi_t = \int_0^t < \delta_s, dW_s > + \int_0^t \beta_s ds \tag{10}
$$

where W_t is a $\mathcal{F}_t$-Brownian motion of dimension k, $(\delta_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$ are bounded $\mathcal{F}_t$-adapted processes that belong respectively to $\mathbb{R}^k$ and to $\mathbb{R}$.

Theorem 2.11 *If ξ_t is an Itô process defined as in (10) satisfying the uniform ellipticity condition: $\exists \lambda \in \mathbb{R}_+^*$ such as $\delta\delta^* \geq \lambda$*
then there exist bounded measurable functions $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by:

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$
$$\begin{aligned} \sigma(t,x) &= \Big(\mathbb{E}\big[\delta_t\delta_t^* | \xi_t = x\big]\Big)^{\frac{1}{2}} \\ b(t,x) &= \mathbb{E}\big[\beta_t | \xi_t = x\big] \end{aligned}$$

such that the following SDE:

$$\begin{aligned} dx_t &= \sigma(t,x_t)dW_t + b(t,x_t)dt \\ x_0 &= 0 \end{aligned}$$

has a weak solution with the same one-dimensional marginals as ξ.

3 Generalities on Local Volatility

3.1 Fokker-Planck Equation

Let us assume that:

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t,S_t)dW_t \tag{11}$$

where r and σ are deterministic functions, σ is usually called the local volatility. Under the local volatility dynamics, option prices satisfy the following PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2(t,S_t)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0 \tag{12}$$

and terminal condition $V(S,T) = C(S,T) = PayOff_T(S)$.
If we consider call options, we would get $V(S,T) = (S-K)_+$. It has been proved that one can obtain a forward PDE for $C(K,T)$ instead of fixing (K,T) and obtaining a backward PDE for $C(S,t)$. To get the Forward PDE equation, one could just differentiate (12) twice with respect to the strike K and then get the same PDE, with variable $\phi = \frac{\partial^2 C}{\partial K^2}$ and terminal condition $\delta(S-K)$. ϕ is the transition density of S and is also the Green function of (12). It follows that ϕ as a function of (K,T) satisfies the Fokker-Planck PDE:

$$\frac{\partial \phi}{\partial T} - \frac{\partial^2}{\partial K^2}\big(\frac{\sigma^2(T,K)}{2}K^2\phi\big) + r(T)\frac{\partial}{\partial K}(K\phi) + r(T)K = 0$$

Now, integrate twice this equation taking into account the boundary conditions, one obtains the Forward Parabolic PDE equation:

$$\frac{\partial C}{\partial T} - \frac{\sigma^2(T,K)}{2}K^2\frac{\partial^2 C}{\partial K^2} + r(T)K\frac{\partial C}{\partial K} = 0 \tag{13}$$

with initial condition $C(K,0) = (S_0 - K)_+$. Hence, one obtains Dupire (1994) equation

$$\sigma^2(T,K) = \frac{\frac{\partial C}{\partial T} + r(T)K\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} \tag{14}$$

Moreover, if one expresses the option price as a function of the forward price, one would write a simpler expression:

$$\sigma^2(T,K,S_0) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

where C is now a function of (F_T, K, T) with $F_T = S_0 \exp\left(\int_0^T r(s)ds\right)$.

3.2 Matching Local and Stochastic Volatilities

A stock price diffusion with a stochastic volatility is one of the following form:

$$\frac{dS_t}{S_t} = r(t)dt + \sqrt{V_t}dW_t \tag{15}$$

where V_t is a stochastic process, solution of an SDE and $r(t)$ is a deterministic function of time. (We do not yet discuss the dependence of the stock price and volatility processes, also called Leverage effect)
One can find a relation between the local volatility and a stochastic volatility. First, one applies Tanaka's formula to the stock price process:

$$\begin{aligned} e^{-\int_0^t r(s)ds}(S_t - x)_+ &= (S_0 - x)_+ - \int_0^t r(u)e^{-\int_0^u r(s)ds}(S_u - x)_+ du \\ &\quad + \int_0^t e^{-\int_0^u r(s)ds}\mathbf{1}_{\{S_u > x\}}dS_u + \frac{1}{2}\int_0^t e^{-\int_0^u r(s)ds}dL_u^x(S) \end{aligned}$$

Assuming that $(e^{-\int_0^t r(s)ds}S_t, t \geq 0)$ is a true martingale, then $(\int_0^t \mathbf{1}_{\{S_u > x\}}d(e^{-\int_0^u r(s)ds}S_u), t \geq 0)$ is a martingale and one gets:

$$\begin{aligned} \mathbb{E}[e^{-\int_0^t r(s)ds}(S_t - x)_+] &= \mathbb{E}[(S_0 - x)_+] + x\int_0^t \mathbb{E}[r(u)e^{-\int_0^u r(s)ds}\mathbf{1}_{\{S_u > x\}}]du \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^t e^{-\int_0^u r(s)ds}dL_u^x(S)\right] \end{aligned}$$

Then, differentiating the previous relation and using Fubini theorem, one obtains:

$$d_t C(t,x) = x\mathbb{E}[r(t)e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\mathbb{E}[e^{-\int_0^t r(s)ds}dL_t^a(S)] \tag{16}$$

where $C(t,x) = \mathbb{E}[e^{-\int_0^t r(s)ds}(S_t - x)_+]$ Using a classical characterization of the local time of any continuous semi-martingale:

$$L_t^x(S) = \lim_{\epsilon \to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{\{x \leq S_s < x + \epsilon\}}d < S, S >_s \tag{17}$$

one gets with a permutation of the differentiation and the expectation:

$$d_tC(t,x) = x\mathbb{E}[r(t)e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t>x\}}]dt + \frac{1}{2}\lim_{\epsilon\to 0}\mathbb{E}[\frac{1}{\epsilon}\mathbf{1}_{\{x\le S_t<x+\epsilon\}}e^{-\int_0^t r(s)ds}V_tS_t^2]dt \tag{18}$$

as a result of $d < S, S >_t = V_tS_t^2 dt$. Now, one may write using conditional expectations and the fact that interest rates are assumed to be deterministic, the following identity:

$$\mathbb{E}[V_tS_t^2\mathbf{1}_{\{x\le S_t<x+\epsilon\}}] = \mathbb{E}\big[\mathbb{E}[V_t|S_t]S_t^2\mathbf{1}_{\{x\le S_t<x+\epsilon\}}\big]$$

From there, one easily obtains:

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[V_tS_t^2\mathbf{1}_{\{x\le S_t<x+\epsilon\}}] &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}\big[\mathbb{E}[V_t|S_t]S_t^2\mathbf{1}_{\{x\le S_t<x+\epsilon\}}\big] \\ &= \mathbb{E}[V_t|S_t = x]x^2q_t(x)\end{aligned}$$

where $q_t(x)$ is the value of the density of S_t in x. Since Breeden and Litzenberger (1978), it is well known that $\frac{\partial^2 C(t,x)}{\partial x^2} = e^{-\int_0^t r(s)ds}q_t(x)$. It is also known that $\frac{\partial C}{\partial x} = -\mathbb{E}[e^{-\int_0^t r(s)ds}\mathbf{1}_{\{S_t>x\}}]$ One finally may write:

$$\frac{\partial C}{\partial t} + xr(t)\frac{\partial C}{\partial x} = \mathbb{E}[V_t|S_t = x]\frac{1}{2}x^2\frac{\partial^2 C}{\partial x^2} \tag{19}$$

Comparing equation (14) and the above equation, one may obtain an equation that relates local and stochastic volatility models

$$\sigma^2(t,x) = \mathbb{E}[V_t|S_t = x] \tag{20}$$

Hence, we have proven that if there exists a local volatility such as the one-dimensional marginals of the stock price with the implied diffusion are the same as the ones of the stock price with the stochastic volatility, then the local volatility satisfies equation (20).

Another way to prove this relation is to apply Gyöngy (1986) result. Since the stock price dynamics with a stochastic volatility given by equation (15) and the ones with the local volatility given by equation (11) must have the same one-dimensional marginals, one can apply Gyöngy Theorem: assuming that there exists $\lambda \in \mathbb{R}_+^*$ such that $S^2V \ge \lambda$ we get the well-known relation between the local and the stochastic volatilities:

$$\sigma(t, S_t = x) = \Big(\mathbb{E}\big[V_t|S_t = x\big]\Big)^{\frac{1}{2}}$$

It is important to notice that Gyöngy gives us the existence of such a diffusion in addition to provide an explicit way to construct it. More generally, assuming just that the volatility process is a general continuous semi-martingale, one can also get the same result, and a justification for the use of local volatility models. Hence, we obtain an illustration of Gyöngy's result in a finance framework.

Moreover, it is shown that one can get the relation (20) without using the Forward PDE equation.

As a first remark, we should notice that if we choose V_t such as $\sqrt{V_t} = \sigma(t, S_t)$, we then obtain another direct proof of equation (14).

As a second remark, we can prove that if $(\widetilde{S}_t = e^{-\int_0^t r(s)ds} S_t), t \geq 0)$ is a strict local martingale (which is studied in Cox and Hobson (2005) who named this market situation a bubble), then

$$\begin{aligned}
\mathbb{E}[\int_0^t \mathbf{1}_{\{S_u > x\}} d\widetilde{S}_u] &= \mathbb{E}[\widetilde{S}_t - S_0] - \mathbb{E}[\int_0^t \mathbf{1}_{\{S_u \leq x\}} d\widetilde{S}_u] \\
&= \mathbb{E}[\widetilde{S}_t - S_0]
\end{aligned}$$

since using Madan and Yor (2006), $(\int_0^t \mathbf{1}_{\{S_u \leq x\}} d(e^{-\int_0^u r(s)ds} S_u), t \geq 0)$ is a square integrable martingale. Hence, defining

$$c_{\widetilde{S}}(t) = \mathbb{E}[S_0 - \widetilde{S}_t]$$

and assuming that $c_{\widetilde{S}}$ is a continuously differentiable function, one obtains an extension of equation (14) that is a generalization to the case of strict local martingales. This equation writes

$$\sigma^2(t, x) = \frac{\frac{\partial C}{\partial t} + x r(t) \frac{\partial C}{\partial x} + c'_{\widetilde{S}}(t)}{\frac{1}{2} x^2 \frac{\partial^2 C}{\partial x^2}}$$

4 Applications to the Heston (1993) model and Extensions

4.1 The Simplest Heston Model

The aim of this paragraph is now to compute the local volatility not by excerpting it from the option prices (see for instance Derman and Kani (1994)) but by applying Gyöngy's theorem.

Among the possible choices of stochastic volatility models, we will consider the simplest one, given by the following SDE:

$$\frac{dS_t}{S_t} = W_t dB_t \tag{21}$$

$$S_0 = 1$$

where (W_t) and (B_t) are two independent one-dimensional Brownian motions starting at 0. We do not consider any drift term in our stock diffusion as we look at the forward price dynamics that are driftless by construction.

To make our discussion a little more general than the model presented in equation (21), we write (21):

$$\frac{dS_t}{S_t} = |W_t| sgn(W_t) dB_t$$

$$S_0 = 1$$

Now we define $\beta_t = \int_0^t sgn(W_s) dB_s$, another Brownian motion which is independent of $(W_t, t \geq 0)$ and consequently of the reflecting Brownian motion $(|W_t|, t \geq 0)$. We get the following model:

$$\frac{dS_t}{S_t} = |W_t| d\beta_t, \qquad S_0 = 1$$

Now this modified form leads itself naturally to the generalization:

$$\frac{dS_t}{S_t} = R_t d\beta_t, \qquad S_0 = 1 \tag{22}$$

where, as in subsection 2.1, (R_t) denotes a Bessel process with dimension δ starting at 0 and (β_t) an independent Brownian motion.

Let us consider a Markovian martingale $(\Sigma_t, t \geq 0)$, which is the unique solution of:

$$\frac{d\Sigma_t}{\Sigma_t} = \sigma(t, \Sigma_t) d\beta_t \tag{23}$$
$$\Sigma_0 = 1$$

for some particular diffusion coefficient $\{\sigma(t,x), t \geq 0, x \in \mathbb{R}_+\}$ which has the same one-dimensional marginal distributions as $(S_t, t \geq 0)$ the solution of (22).

We will now use proposition 2.1 to find σ, the local volatility. We follow the notation in subsection 2.1, and introduce a useful notation:

$$L_t^{(\mu)} = I_t - \mu A_t \tag{24}$$
$$\overset{(Law)}{=} N\sqrt{A_t} - \mu A_t \tag{25}$$

where N is a standard gaussian variable independent of A_t. Next we remark as a consequence of (25) that for any fixed $t \geq 0$:

$$(R_t, L_t^{(\mu)}) \overset{(Law)}{=} (R_t, N\sqrt{A_t} - \mu A_t)$$

and

$$\mathbb{E}\big[R_t^2 | L_t^{(\mu)} = l\big] = \mathbb{E}\big[\mathbb{E}(R_t^2 | N, A_t) | N\sqrt{A_t} - \mu A_t = l\big]$$

Since N is independent of R_t, we obtain

$$\mathbb{E}\big[R_t^2 | L_t^{(\mu)} = l\big] = \mathbb{E}\big[\mathbb{E}(R_t^2 | A_t) | N\sqrt{A_t} - \mu A_t = l\big]$$

From (3), we deduce:

$$\mathbb{E}\big[R_t^2 | L_t^{(\mu)} = l\big] = \big(\frac{2}{t}\big)\mathbb{E}\big[A_t | N\sqrt{A_t} - \mu A_t = l\big] \tag{26}$$

Now, the computation of the expression in (26) is a simple exercise, which we present in the following form:

Lemma 4.1 *Let $X > 0$ be a random variable independent from a standard gaussian variable N. Denote $Y^{(\mu)} = N\sqrt{X} - \mu X$. Then:*
i) for any $f : \mathbb{R}_+ \to \mathbb{R}_+$, Borel function, the following formula holds:

$$\mathbb{E}\big[f(X)|Y^{(\mu)} = z\big] = \frac{h^{(\mu)}(f;z)}{h^{(\mu)}(1;z)}$$

where: $h^{(\mu)}(f;z) = \mathbb{E}\Big[\frac{f(X)}{\sqrt{X}}\exp\Big(-\frac{(z+\mu X)^2}{2X}\Big)\Big]$
ii) in particular, for $f(x) = x$, one can write :

$$\mathbb{E}\big[X|Y^{(\mu)} = z\big] = -\left(\frac{\frac{\partial k}{\partial b}}{k}\right)(\frac{z^2}{2}, \frac{\mu^2}{2}) \tag{27}$$

where $k(a,b) = \mathbb{E}\Big[\frac{1}{\sqrt{X}}\exp\Big(-\big(\frac{a}{X} + bX\big)\Big)\Big]$

The proof of this lemma results from elementary properties of conditioning and is left to the reader.
We now give a formula for $\sigma^2(t,x)$ in terms of the law of $A_t \equiv A_t^{(\delta)}$, by using equation (26) and the above lemma. Indeed, it follows from these results that:

$$\mathbb{E}\big[R_t^2|\ln(S_t) = l\big] = -\frac{2}{t}\frac{\frac{\partial k_\delta^t}{\partial b}\big(\frac{l^2}{2}, \frac{1}{8}\big)}{k_\delta^t\big(\frac{l^2}{2}, \frac{1}{8}\big)}$$

where $k_\delta^t(a,b) = \mathbb{E}\Big[\frac{1}{\sqrt{A_t}}\exp\Big(-\big(\frac{a}{A_t} + bA_t\big)\Big)\Big]$.

Using the scaling property, we have $k_\delta^t(a,b) = \frac{1}{t}k_\delta^1(\frac{a}{t^2}, bt^2)$ which allows us to concentrate on $k^\delta(a,b) \equiv k_\delta^1(a,b)$.
The following formula for the density f_δ of A_1 is borrowed from Biane, Pitman and Yor (2001). Denoting $h = \frac{\delta}{2}$, we have:

$$f_\delta(x) \equiv f_h^\sharp(x) = \frac{2^h}{\Gamma(h)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+h)}{\Gamma(n+1)}\frac{(2n+h)}{\sqrt{2\pi x^3}}\exp\Big(-\frac{(2n+h)^2}{2x}\Big) \tag{28}$$

For $\delta = 2$, $A^{(2)}$, or equivalently $f_2(x) = f_1^\sharp(x)$ enjoys a symmetry property (also shown in Biane, Pitman and Yor (2001)):
For any non-negative measurable function g

$$\mathbb{E}\Big[g(\frac{4}{\pi^2 A^{(2)}})\Big] = \sqrt{\frac{2}{\pi}}\mathbb{E}\Big[\frac{1}{\sqrt{A^{(2)}}}g(A^{(2)})\Big], \tag{29}$$

$$f_1^\sharp(x) = (\frac{2}{\pi x})^{\frac{3}{2}}f_1^\sharp(\frac{4}{\pi^2 x}) \tag{30}$$

and

$$f_1^\sharp(x) = \pi\sum_{n=0}^{\infty}(-1)^n(n+\frac{1}{2})e^{-(n+\frac{1}{2})^2\pi^2\frac{x}{2}} \tag{31}$$

From formula (28), one may compute with the change of variables $a = \frac{\alpha^2}{2}, b = \frac{\beta^2}{2}$

$$\begin{aligned} k^\delta(a,b) &\equiv \mathbb{E}\Big[\frac{1}{\sqrt{A^{(\delta)}}} \exp\Big(-\frac{1}{2}\big(\frac{\alpha^2}{A^{(\delta)}} + \beta^2 A^{(\delta)}\big)\Big)\Big] \\ &= \frac{2^h}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{2\pi}} \int_0^\infty \frac{dx}{x^2} e^{-\frac{1}{2}\left(\frac{\alpha^2+(2n+h)^2}{x} + \beta^2 x\right)} \\ &= \frac{2^h}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left((\alpha^2+(2n+h)^2)x + \frac{\beta^2}{x}\right)} dx \qquad (\star) \end{aligned}$$

Also of importance for us, is the result:

$$\frac{\partial}{\partial b}(k^\delta(a,b)) = \frac{-(2^h)}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-\frac{1}{2}\left(\alpha_n^2 x + \frac{\beta^2}{x}\right)}}{x} dx \tag{32}$$

where $\alpha_n = \sqrt{\alpha^2 + (2n+h)^2}$.
Recall the integral representation for the Mc Donald functions K_ν:

$$K_\nu(z) \equiv K_{-\nu}(z) = \frac{1}{2}\big(\frac{z}{2}\big)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} \exp - \big(t + \frac{z^2}{2t}\big) \tag{33}$$

In particular, we have:

$$K_0(z) = \frac{1}{2} \int_0^\infty \frac{dt}{t} e^{-\left(t + \frac{z^2}{2t}\right)}$$

As a consequence:

$$\int_0^\infty \frac{du}{u} e^{-\frac{1}{2}\left(\alpha^2 u + \frac{\beta^2}{u}\right)} = 2K_0\left(\frac{\alpha\beta}{\sqrt{2}}\right) \tag{34}$$

Now, plugging (34) in (32), we obtain:

$$\frac{\partial}{\partial b}(k^\delta(a,b)) = \frac{-(2^h)}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{2\pi}} 2K_0\left(\frac{\alpha_n\beta}{\sqrt{2}}\right) \tag{35}$$

Likewise, we deduce from (33) that:

$$K_1(z) \equiv K_{-1}(z) = \frac{1}{z} \int_0^\infty dt e^{-\left(t + \frac{z^2}{2t}\right)}$$

which implies

$$\int_0^\infty du e^{-\frac{1}{2}\left(\alpha_n^2 u + \frac{\beta^2}{u}\right)} = \frac{\beta\sqrt{2}}{\alpha_n} K_1\big(\frac{\alpha_n\beta}{\sqrt{2}}\big)$$

Hence, we get as a consequence of $(\star)$:

$$k^\delta(a,b) = \frac{2^h}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{\pi}} \frac{\beta}{\alpha_n} K_1\left(\frac{\alpha_n\beta}{\sqrt{2}}\right) \tag{36}$$

Recalling that $\beta = \sqrt{2b}$ and that $\alpha_n = \sqrt{2a + (2n+h)^2}$, we may now write (36) and (35) as:

$$k^\delta(a,b) = \frac{2^h}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{\pi}} \frac{\sqrt{2b}}{\alpha_n} K_1(\alpha_n \sqrt{b}) \tag{37}$$

$$\frac{\partial}{\partial b}(k^\delta(a,b)) = \frac{-(2^h)}{\Gamma(h)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+h)}{\Gamma(n+1)} \frac{2n+h}{\sqrt{2\pi}} 2K_0(\alpha_n \sqrt{b}) \tag{38}$$

And finally, we obtain the following formula for the local volatility:

$$\sigma^2(t, x = e^l) = -\frac{4}{t} \left(\frac{\frac{\partial k^\delta}{\partial b}}{k^\delta} \right) \left(\frac{l^2}{2t^2}, \frac{t^2}{8} \right) \tag{39}$$

4.2 Adding the Correlation

We now assume a non-zero correlation between the volatility process and the stock price process. This is a common fact in finance called the Leverage Effect and translated by a negative correlation. For a financial understanding of this effect, one can refer for instance to Black (1976), Christie (1982) or Schwert (1989).

Let us define our new model for the stock price dynamics with a Bessel process of dimension δ starting from 0 correlated to the Brownian motion of the stock price process:

$$\frac{dS_t}{S_t} = R_t dW_t \tag{40}$$

$$dR_t^2 = 2R_t dW_t^\sigma + \delta t \tag{41}$$

$$d < W^\sigma, W >_t = \rho dt \tag{42}$$

$$S_0 = 1 \qquad and \qquad R_0 = 0 \tag{43}$$

Then, there exists a Brownian motion β independent of the Bessel process such that $\forall t$:

$$W_t = \rho W_t^\sigma + \sqrt{1-\rho^2} \beta_t$$

Using the previous formula, plugging it in (40) and then inserting (41) in the new (40), one gets:

$$\frac{dS_t}{S_t} = \frac{\rho}{2}(dR_t^2 - \delta dt) + \sqrt{1-\rho^2} R_t d\beta_t \tag{44}$$

Then using Itô formula applied to $f(x) = ln(x)$

$$d\ln(S_t) = \frac{dS_t}{S_t} - \frac{1}{2} R_t^2 dt$$

one obtains:

$$\ln(S_t) = \frac{\rho}{2}(R_t^2 - \delta t) + \sqrt{1-\rho^2} \int_0^t R_s d\beta_s - \frac{1}{2} \int_0^t R_s^2 ds \tag{45}$$

Let us consider as in subsection 5.1, $L_t^{(\mu)} = \int_0^t R_s d\beta_s - \mu \int_0^t R_s^2 ds$ (we are especially interested in the case $\mu = \frac{1}{2\sqrt{1-\rho^2}}$). Since R and β are independent, we shall use the same notation as above. Particularly, A_t and I_t will refer to the quantities defined in subsection 2.1.
Now, equation (45) can be rewritten as follows:

$$\ln(S_t) = \frac{\rho}{2}(R_t^2 - \delta t) + \sqrt{1-\rho^2} L_t^{(\frac{1}{2\sqrt{1-\rho^2}})} \tag{46}$$

Since we wish to evaluate the local volatility $\mathbb{E}\big[R_t^2 | \ln(S_t) = l\big]$, we will try to compute more generally the following quantity:

$$\mathbb{E}\big[R_t^2 | m R_t^2 + L_t^{(\mu)} = l\big] \tag{47}$$

where m is a real constant.

Remark 4.2 *We immediately see that if we take $m = 0$, i.e $\rho = 0$, we are back to the previous paragraph setting.*

First, we see that equation (26) is easily extended to the case with correlation and we obtain:

$$\mathbb{E}\big[R_t^2 | m R_t^2 + L_t^{(\mu)} = l\big] = \frac{2}{t}\mathbb{E}\big[A_t | m R_t^2 + L_t^{(\mu)} = l\big] \tag{48}$$

Before extending Lemma 4.1, one must recall that for any $t \geq 0$:

$$(R_t, A_t, L_t^{(\mu)}) \stackrel{(Law)}{=} (R_t, A_t, N\sqrt{A_t} - \mu A_t)$$

where N is a standard gaussian variable independent of R_t and A_t. The following simple result will be helpful for the remaining of the paper:

Lemma 4.3 *Let $X > 0$ and $Z \geq 0$ independent from a standard gaussian variable N. Denote $Y^{(\mu)} = N\sqrt{X} - \mu X$. Then:*
i) for any Borel function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$, real number m we have the formula:

$$\mathbb{E}\big[f(X,Z) | mZ + Y^{(\mu)} = z\big] = \frac{a^{(\mu,m)}(f;z)}{a^{(\mu,m)}(1;z)}$$

where: $a^{(\mu,m)}(f;z) = \mathbb{E}\left[\frac{f(X,Z)}{\sqrt{X}} \exp\left(- \frac{(z+\mu X - mZ)^2}{2X}\right)\right]$
ii) in particular, for $f(x,y) = x$, we obtain:

$$\mathbb{E}\big[X | mZ + Y^{(\mu)} = z\big] = -\frac{1}{\mu\sqrt{2}}\left(\frac{\frac{\partial \alpha}{\partial b}}{\alpha}\right)\left(\frac{z}{\sqrt{2}}, \frac{\mu}{\sqrt{2}}, \frac{m}{\sqrt{2}}\right) \tag{49}$$

where $\alpha(a,b,c) = \mathbb{E}\left[\frac{1}{\sqrt{X}} \exp\left(- \left(\frac{(a-cZ)^2}{X} + b^2 X + bcZ\right)\right)\right]$

The other fundamental result we now need, is the joint density of $(R_t^2, \int_0^t R_s^2 ds)_{t\geq 0}$.

Theorem 4.4 *The joint distribution g_t of $(R_t^2, \int_0^t R_s^2 ds)$ is given by:*

$$g_t(x,y) = \frac{1}{\sqrt{2\pi}\Gamma(\frac{\delta}{2})} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^{j+\frac{\delta}{2}-1} y^{-\frac{j}{2}-\frac{\delta}{4}-1} f_t^j(x,y) \tag{50}$$

where f_t^j is defined by

$$f_t^j(x,y) = \sum_{k=0}^{\infty} \frac{(j+\frac{\delta}{2})_k}{k!} e^{-\frac{1}{4y}[2(k+j+\frac{\delta}{4})t+\frac{x}{2}]^2} D_{\frac{\delta}{2}+j+1}\left(\frac{2(k+j+\frac{\delta}{4})t+\frac{x}{2}}{\sqrt{y}}\right) \tag{51}$$

$D_\nu(\xi)$ is a parabolic cylinder function and $(\nu)_k$ the Pochhammer's symbol defined by $(\nu)_k \equiv \nu(\nu+1)...(\nu+k-1) = \Gamma(\nu+k)/\Gamma(\nu)$

Proof : See Ghomrasni (2004) who evaluates the Laplace transform of (2) in order to get the density function. ∎

For the definition and properties on the parabolic cylinder functions, we refer to Gradshteyn and Ryzhik (2000).
Let us define α_t in the following form:

$$\alpha_t(a,b,c) = \mathbb{E}\left[\frac{1}{\sqrt{A_t}} \exp\left(-\left(\frac{(a-cR_t^2)^2}{A_t} + b^2 A_t + bcR_t^2\right)\right)\right] \tag{52}$$

Unfortunately, there is no more scaling property as in the zero-correlation case and we may not rewrite α_t as a function of t and α_1. One can then compute the local volatility $\sigma^{(\rho)}$ by noticing that in the case of particular interest for us, the parameters are defined as follows:

$$m = \frac{\rho}{2\sqrt{1-\rho^2}} \quad \text{and} \quad z = \frac{l+\frac{\rho}{2}\delta t}{\sqrt{1-\rho^2}} \quad \text{and} \quad \mu = \frac{1}{2\sqrt{1-\rho^2}}$$

We then obtain

$$\sigma^{(\rho)}(t, x=e^l) = -\sqrt{2(1-\rho^2)}\left(\frac{\frac{\partial \alpha_t}{\partial b}}{\alpha_t}\right)\left(\frac{l+\frac{\rho}{2}\delta t}{\sqrt{2(1-\rho^2)}}, \frac{1}{\sqrt{8(1-\rho^2)}}, \frac{\rho}{\sqrt{8(1-\rho^2)}}\right) \tag{53}$$

4.3 From a Bessel Volatility process to the Heston Model

The Heston (1993) model for representing a stochastic volatility process is a particular case of the Cox, Ingersoll and Ross (1985) stochastic process, of the form:

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t \tag{54}$$

with initial condition $V_0 = v_0$

Actually, it is possible to find out deterministic space and time changes such as the law of the Heston SDE solution and the Time-Space transformed Bessel Process are the same.

Proposition 4.5 *For every Heston SDE solution, there exist a Bessel process and two deterministic functions f and g with g increasing such as:*

$$V_t = f(t) \times R^2_{g(t)}$$

where R denotes a Bessel Process of dimension $\delta = \frac{4\kappa\theta}{\eta^2}$ starting from $\sqrt{v_0}$ at time $t = 0$ and f and g are defined by:

$$f(t) = e^{-\kappa t}$$
$$g(t) = \frac{\eta^2}{4\kappa}(e^{\kappa t} - 1)$$

Proof : It is just an application of Lemma 2.4. ■

One may now apply the results of the previous sections using the time and space transformations presented in the previous paragraph

Proposition 4.6 *Let us consider the following stochastic volatility model:*

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t} d\beta_t, \qquad S_{\{t=0\}} = S_0 \\
v_t &= \frac{\eta^2}{4} e^{2\kappa t} V_t, \qquad v_0 = \frac{\eta^2}{4} V_0 \\
dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dW_t \\
d < \beta_., W_. >_t &= \rho dt
\end{aligned}$$

where β_t is a Brownian motion and V_t is an Heston process as defined above. Then the local volatility $\tilde{\sigma}$ that gives us the expected mimicking properties, satisfies the following equation:

$$\tilde{\sigma}(t, x) = \frac{\eta^2}{4} e^{\kappa t} \sigma\Big(\frac{\eta^2}{4\kappa}(e^{\kappa t} - 1), \frac{x}{s_0}\Big) \tag{55}$$

where $\sigma^2(t, x) = \mathbb{E}[R_t^2 | \exp(I_t - \frac{1}{2} A_t) = x]$ and R_t is a Bessel Process of dimension $\delta = \frac{4\kappa\theta}{\eta^2}$ starting from V_0.

Proof : First, one has the Gyöngy volatility formula:

$$\tilde{\sigma}^2(t, x) = \frac{\eta^2}{4} e^{2\kappa t} \mathbb{E}\big[V_t | S_t = x\big] \tag{56}$$

Then using Lemma 2.4, one easily obtains the result. ■

Remark 4.7 *Let us note that we only have closed-form formulas in cases where $V_0 = 0$ and that otherwise we have to go through Laplace transform inversion techniques.*

One can propose a general framework for constructing stochastic volatility models based on Bessel processes. Local volatilities can be computed through the proposition below whose proof is left to the reader.

Proposition 4.8 *Let us consider the following stochastic volatility model:*

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t}d\beta_t, \qquad S_{\{t=0\}} = S_0 \\
v_t &= \frac{g'(t)}{f(t)}V_t, \qquad v_0 = \frac{g'(0)}{f(0)}V_0 \\
dV_t &= \left(\delta f(t)g'(t) + \frac{f'(t)}{f(t)}V_t\right)dt + \sqrt{f(t)}g'(t)\sqrt{V_t}dW_t \\
d < \beta_., W_. >_t &= \rho dt
\end{aligned}$$

where β_t and W_t are Brownian motions, f is a positive continuously differentiable function and g an increasing $\mathcal{C}^1$ function.
Then the local volatility $\tilde{\sigma}$ that gives us the expected mimicking properties, satisfies the following equation:

$$\tilde{\sigma}(t,x) = g'(t)\sigma\big(g(t), \frac{x}{S_0}\big) \tag{57}$$

where $\sigma^2(t,x) = \mathbb{E}[R_t^2|\exp(I_t - \frac{1}{2}A_t) = x]$ and R_t is a Bessel Process of dimension δ starting from V_0.

5 Pricing Equity Derivatives under Stochastic Interest Rates

5.1 A Local Volatility Framework

With the growth of hybrid products, it has been necessary to take properly into account the stochasticity of interest rates in FX or Equity models in a way that makes the equity volatility surface calibration easy at a given interest rate parametrization. It has been now a while that people have been considering interest rates as stochastic for long-dated Equity or FX options, but they have not been thinking about it in terms of calibration issues. Besides, according to the interest rates part of an equity - interest rates hybrid product for example, the instruments on which the interest rates model will be calibrated are different; hence it becomes necessary to parameterize the volatility surface efficiently. For most of hybrid products, no forward volatility dependence is involved and then a local volatility framework is sufficient. Let us now consider a local volatility model with stochastic interest rates:

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, S_t)dW_t$$

where r_t is a stochastic process and σ a deterministic function.
Now, we can observe that equation (18) is still valid under stochastic rates and we may then write

$$d_tC(t,x) = x\mathbb{E}[r_t e^{-\int_0^t r_s ds}\mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2}\lim_{\epsilon\to 0}\mathbb{E}[\frac{1}{\epsilon}\mathbf{1}_{\{x\le S_t < x+\epsilon\}}e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2]dt$$

The second term of the right-hand side may be written as follows

$$\mathbb{E}[e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}] = \mathbb{E}\big[\mathbb{E}[e^{-\int_0^t r_s ds}|S_t]\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}\big]$$

and then we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[e^{-\int_0^t r_s ds}\sigma^2(t,S_t)S_t^2\mathbf{1}_{\{x\leq S_t<x+\epsilon\}}] = x^2\sigma^2(t,x)q_t(x)\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]$$

where $q_t(x)$ is the value of the density of S_t in x. It is easily shown as well that

$$\frac{\partial^2 C}{\partial x^2} = q_t(x)\mathbb{E}[e^{-\int_0^t r_s ds}|S_t = x]$$

Let us now define the t-forward measure $\mathbb{Q}^t$ (see Geman (1989), Jamshidian (1989)) by

$$\frac{d\mathbb{Q}^t}{d\mathbb{Q}} = \frac{e^{-\int_0^t r_s ds}}{B(0,t)} \qquad \text{where} \qquad B(0,t) = \mathbb{E}[e^{-\int_0^t r_s ds}]$$

Hence, we finally obtain an extension of Dupire (1994)'s formula :

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} - xB(0,t)\mathbb{E}^t[r_t\mathbf{1}_{\{S_t>x\}}]}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}$$

Under a T-forward measure for $T \geq t$, one has

$$\mathbb{E}^T[r_T|\mathcal{F}_t] = f(t,T)$$

where $f(t,T)$ is the instantaneous forward rate. To conclude this subsection, we can first notice that this slight extension of Dupire equation may be also written

$$\sigma^2(t,x) = \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x} - xB(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t>x\}})}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}} \tag{58}$$

We then assume that it is possible to extract from markets prices the quantities $\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t>x\}})$ (i.e. there exist tradeable assets from which we could obtain these covariances) in order to add stochastic interest rates to the usual local volatility framework. For the remainder of the paper, we denote this assumption the (HC)-Hypothesis that stands for Hybrid Correlation hypothesis. Under this market hypothesis, one is able to calibrate a local volatility surface with stochastic interest rates implied by the derivatives' market prices.

5.2 Mimicking Stochastic Volatility Models

In this subsection, we consider the case of a stochastic volatility model with stochastic interest rates and see how it is possible to connect it to a local volatility framework. Let us consider the following diffusion

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t}dW_t$$

with V_t a stochastic process and let us use equation (18) in order to exhibit a new mimicking property:

$$d_t C(t,x) = x\mathbb{E}[r_t e^{-\int_0^t r_s ds} \mathbf{1}_{\{S_t > x\}}]dt + \frac{1}{2} \lim_{\epsilon \to 0} \mathbb{E}[\frac{1}{\epsilon} \mathbf{1}_{\{x \le S_t < x+\epsilon\}} e^{-\int_0^t r_s ds} V_t S_t^2]dt$$

Then,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}[e^{-\int_0^t r_s ds} V_t S_t^2 \mathbf{1}_{\{x \le S_t < x+\epsilon\}}] &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\big[\mathbb{E}[V_t e^{-\int_0^t r_s ds} | S_t] S_t^2 \mathbf{1}_{\{x \le S_t < x+\epsilon\}}\big] \\
&= x^2 q_t(x) \mathbb{E}[V_t e^{-\int_0^t r_s ds} | S_t = x] \\
&= x^2 \frac{\partial^2 C}{\partial x^2} \frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds} | S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds} | S_t = x]}
\end{aligned}$$

Hence, we obtain

$$\frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds} | S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds} | S_t = x]} = \frac{\frac{\partial C}{\partial t} + x f(0,t) \frac{\partial C}{\partial x} - x B(0,t) \mathbb{C}ov^t(r_t; \mathbf{1}_{\{S_t > x\}})}{\frac{x^2}{2} \frac{\partial^2 C}{\partial x^2}}$$

Spot Mimicking Property Finally, if there exists a stochastic process, solution of the following SDE

$$\frac{dX_t}{X_t} = r_t dt + \sigma(t, X_t) dW_t$$

such that the one-dimensional marginals of the triple $(r_t, \int_0^t r_s ds, X_t)$ are the same as $(r_t, \int_0^t r_s ds, S_t)$, then by identification one must have

$$\sigma^2(t,x) = \frac{\mathbb{E}[V_t e^{-\int_0^t r_s ds} | S_t = x]}{\mathbb{E}[e^{-\int_0^t r_s ds} | S_t = x]}$$

The existence is easily proven in the cases where $(r_t, t \ge 0)$ is a Markovian diffusion. Hence, we exhibit a strong mimicking property since we obtained an explicit way to construct a local volatility surface.

Remark 5.1 *We may notice that if interest rates are deterministic, we recover the well-known formula (20).*

Forward Mimicking Property Let us now write a Forward mimicking property by applying Gyöngy's result to match the one dimensional marginals of a stochastic volatility model and of a local volatility one:
If one defines $F_t^{(1)} = S_t e^{-\int_0^t r_s ds}$ and $F_t^{(2)} = X_t e^{-\int_0^t r_s ds}$ where S and X are defined above, we obtain the existence of diffusions $Y_t^{(1)}$ and $Y_t^{(2)}$ solutions of

$$\frac{dY_t^{(i)}}{Y_t^{(i)}} = \Sigma_{(i)}(t, Y_t^{(i)}) dW_t$$

for $i = 1, 2$ such as

$$\begin{aligned}
\Sigma^2_{(1)}(t,x) &= \mathbb{E}[V_t | S_t = xe^{\int_0^t r_s ds}] \\
\Sigma^2_{(2)}(t,x) &= \mathbb{E}[\sigma^2(t, xe^{-\int_0^t r_s ds}) | X_t = xe^{\int_0^t r_s ds}]
\end{aligned}$$

Since the one-dimensional marginals of $F_t^{(1)}$ and $F_t^{(2)}$ must be equal, one obtains

$$\mathbb{E}[V_t | S_t = xe^{\int_0^t r_s ds}] = \mathbb{E}[\sigma^2(t, xe^{-\int_0^t r_s ds}) | X_t = xe^{\int_0^t r_s ds}] \tag{59}$$

We consequently obtain an implicit way to construct a local volatility surface we say that this relation is weak in the sense that it is a weak mimicking distribution property which is involved in the above relation.

5.3 From a Deterministic Interest Rates Framework to a Stochastic one

Going from a framework to another is valuable for calibration issues. Let us assume, for instance that a model has been calibrated with deterministic interest rates and that one wants to recalibrate the same model assuming stochastic interest rates. Let us introduce some notation to define the different kinds of frameworks we will go through in this subsection.

Notation
LV stands for Local Volatility, **SV** stands for Stochastic Volatility, **DIR** stands for Deterministic Interest Rates and **SIR** stands for Stochastic Interest Rates

From DIR-LV to SIR-LV Let us first consider the local volatility case. Under deterministic interest rates, the stock price dynamics are driven by the equation

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t, S_t)dW_t$$

while under stochastic interest rates it would be

$$\frac{dS_t}{S_t} = r_t dt + \overline{\sigma}(t, S_t)dW_t$$

and we know that both local volatility functions solve the following implied equations:

$$\begin{aligned}
\sigma^2(t,x) &= \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x}}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}} \\
\overline{\sigma}^2(t,x) &= \frac{\frac{\partial C}{\partial t} + xf(0,t)\frac{\partial C}{\partial x} - xB(0,t)\mathbb{C}ov^t(r_t; \mathbf{1}_{\{S_t > x\}})}{\frac{x^2}{2}\frac{\partial^2 C}{\partial x^2}}
\end{aligned}$$

where $f(0,t) = r(t)$.

Now, if the prices involved in the estimation of the local volatility surfaces are observed on markets and respect the (HC)-Hypothesis, one may write

$$\sigma^2(t,x) - \overline{\sigma}^2(t,x) = \frac{2B(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t>x\}})}{x\frac{\partial^2 C}{\partial x^2}} \tag{60}$$

From DIR-SV to SIR-SV If we assume that a general Itô process drives the volatility we will write

$$\begin{aligned}
\frac{dS_t^{(1)}}{S_t^{(1)}} &= r(t)dt + \sqrt{V_t^{(1)}}dW_t \\
\frac{dS_t^{(2)}}{S_t^{(2)}} &= r_t dt + \sqrt{V_t^{(2)}}dW_t
\end{aligned}$$

and then, if $S^{(1)}$ and $S^{(2)}$ have the same one-dimensional marginals, we obtain the following relation to relate $V^{(1)}$ to $V^{(2)}$:

$$\mathbb{E}[V_t^{(1)}|S_t^{(1)}=x] - \frac{\mathbb{E}[V_t^{(2)}e^{-\int_0^t r_s ds}|S_t^{(2)}=x]}{\mathbb{E}[e^{-\int_0^t r_s ds}|S_t^{(2)}=x]} = \frac{2B(0,t)\mathbb{C}ov^t(r_t;\mathbf{1}_{\{S_t^{(2)}>x\}})}{x\frac{\partial^2 C}{\partial x^2}} \tag{61}$$

From SIR-SV to DIR-LV Let us now specify a Heath Jarrow and Morton (1992) diffusion for the interest rate model and see precisely how one could extract, using Gyöngy's result, the volatility of the forward contract under deterministic interest rates from the volatility of the forward contract under stochastic interest rates. Let us recall that in a standard HJM framework, the instantaneous forward rate follows

$$df(t,T) = \Big(\sigma(t,T)\int_t^T \sigma(t,u)du\Big)dt + \sigma(t,T)dW_t^r$$

where $\sigma(t,T)$ is a stochastic process adapted to its canonical filtration and where the price satisfies

$$B(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big)$$

By definition $r_t = f(t,t)$ and then we obtain

$$\begin{aligned}
\frac{dB(t,T)}{B(t,T)} &= r_t dt - \sigma_B(t,T)dW_t^r \\
\sigma_B(t,T) &= \int_t^T \sigma(t,u)du
\end{aligned}$$

For our purpose, let us consider a general model

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t} dW_t$$

then recall the price of the T-forward contract written on S

$$F_t^T = \frac{S_t}{B(t,T)}$$

where we assume $d < W, W^r >_t = \rho dt$. We are now able to write the dynamics of F_t^T under $\mathbb{Q}$ the risk-neutral measure:

$$\frac{dF_t^T}{F_t^T} = \frac{dS_t}{S_t} - \frac{d < S., B(\cdot,T) >_t}{S_t B(t,T)} - \left(\frac{dB(t,T)}{B(t,T)} - \frac{d < B(\cdot,T) >_t}{B^2(t,T)} \right)$$

If we introduce the T-forward probability measure as above by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{e^{-\int_0^T r_s ds}}{B(0,T)}$$

we explain the dynamics of F_t^T under $\mathbb{Q}^T$

$$\frac{dF_t^T}{F_t^T} = \sqrt{V_t} d\widetilde{W}_t + \sigma_B(t,T) d\widetilde{W}_t^r$$

where $\widetilde{W}$ and $\widetilde{W}^r$ are Brownian motions under $\mathbb{Q}^T$ such as $d < \widetilde{W}, \widetilde{W}^r >_t = \rho dt$. We now apply Theorem 2.11 and obtain the existence of a process $\widetilde{F}_t^T$ solution of an inhomogeneous Markovian stochastic differential equation

$$\frac{d\widetilde{F}_t^T}{\widetilde{F}_t^T} = \Sigma_T(t, \widetilde{F}_t^T) d\beta_t$$

where β is a Brownian motion and

$$\Sigma_T^2(t,x) = \mathbb{E}^T[V_t + 2\rho\sqrt{V_t}\sigma_B(t,T) + \sigma_B^2(t,T) | S_t = xB(t,T)]$$

If we consider a local volatility model with deterministic interest rates as follows

$$\frac{dS_t}{S_t} = f(0,t)dt + \sigma(t, S_t) d\gamma_t$$

the dynamics of the T-forward contract then becomes

$$\frac{dF_t^T}{F_t^T} = \sigma(t, F_t^T e^{-\int_t^T f(0,s)ds}) d\gamma_t$$

and Gyöngy's result enables us to conclude that

$$\Sigma_T(t,x) = \sigma(t, xe^{-\int_t^T f(0,s)ds})$$

Hence, we have proven a new relation that links a local volatility framework with deterministic interest rates to a stochastic volatility one with stochastic interest rates, namely

$$\sigma^2(t,x) = \mathbb{E}^T[V_t + 2\rho\sqrt{V_t}\sigma_B(t,T) + \sigma_B^2(t,T)|S_t = xB(t,T)e^{\int_t^T f(0,s)ds}] \quad (62)$$

An illustration of this formula can be found for a Black and Scholes (1973) framework with random rates for example in Hull and White (1994).

6 Conclusion

This paper recalls well-known results on local volatility and establishes links to stochastic volatility through the powerful theorems of Krylov and Gyöngy. These general results are then illustrated with explicit computations of local volatility in different stochastic volatility models where the volatility process is a time-space transformation of Bessel processes. In this framework, we show the impact of the stock-volatility correlation on the local volatility surface.
The local volatility extracted from a stochastic volatility model allows us to get a precise idea of the skew generated by a stochastic volatility model. Hence, an important theoretical and numerical advantage of generating a local volatility surface from a stochastic volatility rather than from market option prices is the stability and the meaningfulness of the surface. Indeed, the local volatility surface constructed with the Forward PDE equation is known to be completely unstable whereas as one can see the one built from a stochastic volatility is really smooth.
With the growth of hybrid products, it has been important to seriously consider the issue of volatility calibration under stochastic interest rates and that is the reason why we exhibit different relations between local volatilities, stochastic volatilities and derivative prices. It is shown that Dupire (1994) and Derman and Kani (1998) formulas can easily be extended and that it is possible to relate any continuous stochastic volatility model with stochastic interest rates to a local volatility one with deterministic interest rates. By extending the local volatility formula to a stochastic rates framework, it is observed that a market premium for the hybrid correlation risk is to be implied for the construction of the local volatility surface, which can be performed under the (HC)-Hypothesis as at some point a market premium for the volatility risk is to be taken into account.
A remaining interesting question is the existence of a local volatility diffusion with a general Ito interest rates process framework such that the joint law of instantaneous rate, the discount factor and the stock price is the same as the one in a stochastic volatility framework.

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